Math Preliminaries / Notation

"series expansions / asymptotic approx's / self-consistency"

simple example: $\epsilon x^2 + x - 1 = 0$

$\Rightarrow$ positive root: $x = \dfrac{-1 + \sqrt{1+4\epsilon}}{2\epsilon} = F(\epsilon)$

e.g. $\epsilon \to 0$

$\Rightarrow$ Taylor series: $x \approx F(0) + F'(0)\epsilon + \ldots$

$= \underbrace{(1)}_{\text{"leading order"}} + \underbrace{(-1)}_{\text{"next order"}}\epsilon + \ldots$

$\Rightarrow \quad x \approx F(0)$ if $F'(0)\epsilon \ll F(0)$

$\Rightarrow$ if $\epsilon \ll \epsilon^* \equiv \dfrac{F(0)}{F'(0)} = 1$

$\Rightarrow$ can write this as $x \approx 1 \quad (\epsilon \ll 1)$

* can also do same thing from eq itself
"dominant balance"

small

Step 1: guess $\cancel{\epsilon x^2} + x - 1 = 0 \Rightarrow x = 1$

"leading order approx"

Step 2: check whether approx is self-consistent

$$\Rightarrow \epsilon x^2 \approx \epsilon (1)^2 = \epsilon \ll 1, x = 1$$

$\Rightarrow$ self consistent if $\boxed{\epsilon \ll 1}$

$\Rightarrow$ can tell us when approx breaks down:

e.g. $\epsilon A x^2 + x - 1 = 0 \Rightarrow x \approx 1$

$\Rightarrow$ self-consistent when $\boxed{\epsilon \ll Y_A}$

compare to: $\lim_{\epsilon \to 0} F(\epsilon) = 1$ $\quad \lim_{\epsilon \to 0} F_1(\epsilon) = 1$

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tells us when wrong: $x \ll \epsilon x^2, -1 \Rightarrow x = \cancel{\frac{1}{\sqrt{\epsilon}} \ll 1}$

* step 1 $\quad x = 1 + \delta$ correction term

step 2: substitute into $\epsilon x^2 + x - 1 = 0$

$$\epsilon(1+\delta)^2 + (1+\delta) - 1 = 0$$

$$\Downarrow$$

$$\epsilon(1+2\delta) + (1+\delta) - 1 = 0$$

$$\Rightarrow \delta = \frac{-\epsilon}{1+2\epsilon} \approx -\epsilon$$

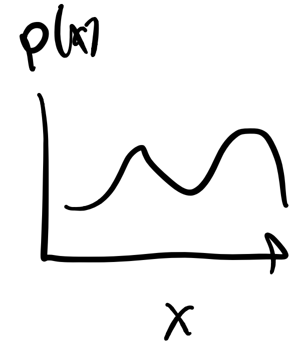
$\Rightarrow$ e.g. $\epsilon x^5 + x - 1 = 0$

Useful in practical contexts:

w/ all possible choices of ϵ

$\Rightarrow$ most are $\boxed{\ll 1}$, $\boxed{\gg 1}$

Probability



① Random variable, $\hat{x}$, distributed according to $p(x)$

$\Rightarrow$ notation $x \sim p(x)$

② mean (expected value) $\langle x \rangle \equiv E[x] \equiv \int x p(x) dx$

③ variance: $\mathrm{Var}(x) \equiv \langle x^2 \rangle - \langle x \rangle^2$

④ Common statistical dist'ns:

$n \sim \mathrm{Binomial}(N, p) \quad \left[P(n) = \binom{N}{n} p^n (1-p)^{N-n} \right]$

$n \sim \mathrm{Poisson}(\langle n \rangle) = \lim_{\substack{N \to \infty \\ p \to 0 \\ \text{fixed } \langle n \rangle = \lambda}} \mathrm{Binom}(N, p) \quad \left[P(n) = \frac{\lambda^n}{n!} e^{-\lambda} \right]$

$x \sim \mathrm{Gaussian}(\mu, \sigma^2) \quad \left[p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right]$

"Normal"

$\Rightarrow$ wikipedia is your friend here!

③ Joint dist'n: $p(x,y) = $ prob $\hat{x}=x$ & $\hat{y}=y$

$$p(x) = \int p(x,y)\,dy \quad \text{(marginalization)}$$

* conditional probability $p(x|y) \equiv \dfrac{p(x,y)}{p(y)}$ "prob $\hat{x}=x$ given $\hat{y}=y$"

* statistical independence: $p(x,y) = p(x)\,p(y)$

-or- $p(x|y) = p(x)$

$\Rightarrow$ for ind. R.V. $\mathrm{Var}(x+y) = \mathrm{Var}(x) + \mathrm{Var}(y)$

④ Moment generating function

$$H_x(z) = \langle e^{-zx} \rangle = \int e^{-zx}\,p(x)\,dx$$ "Laplace transform"

$\Rightarrow H_x(z) \Leftrightarrow p(x)$

hard!

e.g. normal R.V.

$$H_x(z) = e^{-\boxed{\mu} z + \frac{1}{2}\boxed{\sigma^2} z^2}$$

e.g. Poisson: $H_n(z) = e^{-\lambda(1-e^{-z})}$

$$H_x(z) = \int e^{-zx} p(x)\,dx$$

$$\simeq \int \left(1 - zx + \frac{1}{2} z x^2 + \dots\right) p(x)\,dx$$

$$\simeq 1 - z\langle x\rangle + \frac{1}{2} z^2 \langle x^2\rangle + \dots$$

e.g. $H_x(z) = e^{-\mu z + \frac{1}{2}\sigma^2 z^2} \simeq 1 - \mu z + \frac{1}{2} z^2 \sigma^2 + \frac{1}{2} z^2 \mu^2$

$\Rightarrow \langle x\rangle = \mu$, $\langle x^2\rangle = \sigma^2 + \mu^2$

$\Rightarrow \mathrm{Var}(x) = \langle x^2\rangle - \langle x\rangle^2 = \sigma^2$

shortcut: if $H_x(z) = \exp[Q(z)]$

$$\rightarrow Q(z) \approx 0 - \langle x \rangle z + \frac{1}{2} \mathrm{Var}(x) z^2 + \ldots$$

$\Rightarrow$ Payoff for independent R.Vs:

$$H_{x+y}(z) = \langle e^{-z(x+y)} \rangle = \langle e^{-zx} \cdot e^{-zy} \rangle$$

$$= \langle e^{-zx} \rangle \langle e^{-zy} \rangle$$

$$= H_x(z) H_y(z)$$

$$\Rightarrow \quad x = x_1 + x_2 \quad \Rightarrow \quad p(x) = \int dx_1 dx_2 \, p(x_1) p(x_2) \times \delta(x - x_1 - x_2)$$

$$\Rightarrow \quad H_{x_1+x_2}(z) = \exp\left[-\mu_1 z + \frac{1}{2}\sigma_1^2 z^2\right] \times \exp\left[-\mu_2 z + \frac{1}{2}\sigma_2^2 z^2\right]$$

$$= \exp\left[-(\mu_1+\mu_2) z + \frac{1}{2}(\sigma_1^2+\sigma_2^2) z^2\right]$$

central limit thearem

$X_1, X_2, \dots, X_n$ independent

$$\Rightarrow \text{then} \quad \sum_{i=1}^{n} x_i \rightarrow \text{Gaussian}\left(n\langle x\rangle, n\mathrm{Var}(x)\right)$$

$$\Rightarrow \quad \frac{1}{n}\sum_{i=1}^{n} x_i \sim \text{Gaussian}\left(\langle x\rangle, \frac{\mathrm{Var}(x)}{n}\right)$$

$$\approx \langle x\rangle \pm \frac{\mathrm{Var}(x)}{n}$$