Analytical Approximations for Gaussian Integrals

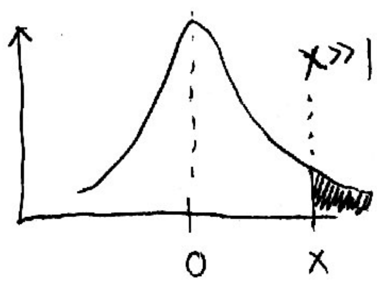
We'll often need to evaluate integrals of the form:

$$I(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dy \qquad \text{for} \quad 0 \leq x \leq \infty.$$

$\Rightarrow$ for $x \ll 1$, we have $I(x) \approx \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy = \frac{1}{2}$

(since $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy = 1$ is a normalized probability distn)

$\Rightarrow$ for $x \gg 1$, things are a bit trickier. Progress comes from realizing that for large x, the argument of the exponential is big for all y, and will be dominated by y values "close to x."



Motivated by this intuition, we'll define $y \equiv x + u$ and change variables to u in integral:

$$I(x) = \int_0^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} - xu - \frac{u^2}{2}}\, du$$

Our intuition is that this integral will be dominated by $u \ll 1$. To make this intuition precise, let's split the integral into two regions:

$$I(x) = \int_0^{u^*} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} - xu - \frac{u^2}{2}} du + \underbrace{\int_{u^*}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2} - xu - \frac{u^2}{2}} du}_{I(x+u^*)}$$

for some value u^* (with the idea that $u^* \ll 1$, but bigger than the u-values that dominate $I(x)$)

$\Rightarrow$ if $u^* \ll 1$, then $e^{-u^2/2} \approx 1$ for $0 \leq u \leq u^*$

(note, same will not be true of e^{-xu} since x is large)

$$\Rightarrow I(x) \approx \int_0^{u^*} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} e^{-xu} du + I(x+u^*)$$

$$\approx \frac{1}{\sqrt{2\pi}} \frac{e^{-x^2/2}}{x} \left(1 - e^{-xu^*}\right) + I(x+u^*)$$

Now we can see that by choosing u^* to be larger than $\sim \mathcal{O}\left(\frac{1}{x}\right) \rightarrow$ e.g. $u^* = \frac{1}{\sqrt{x}}, \frac{1}{x^{1/3}}$, etc.

then:

$$I(x) \approx \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2} + I\left(x + \frac{1}{\sqrt{x}}\right) \quad \text{when } x \gg 1$$

Finally, we'll solve this equation by guessing that

$$I\left(x + \frac{1}{\sqrt{x}}\right) \ll \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}$$

$\Rightarrow$ if so, then $I(x) \approx \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}$

$$\Rightarrow I\left(x + \frac{1}{\sqrt{x}}\right) = \frac{1}{\sqrt{2\pi}} \left(\frac{1}{x + \frac{1}{\sqrt{x}}}\right) e^{-x^2/2 - x^{1/2} - \frac{1}{2x}}$$

$$\approx \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2} \left[\frac{e^{-x^{1/2}}}{(1 + x^{-3/2})}\right]$$

$$\ll \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2} \quad \checkmark \text{ as assumed.}$$

thus, we have:

$$\boxed{I(x) \equiv \int_x^\infty \frac{1}{\sqrt{2\pi}} e^{-y^2/2}\, dy \sim \begin{cases} \frac{1}{2} & \text{for } x \ll 1 \\ \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2} & \text{for } x \gg 1 \end{cases}}$$

In many cases, we'll also want to invert this function to find the value of x that produces a given value of $I(x)$.

(1) For $I(x) \approx \frac{1}{2}$ inversion is easy: $x \approx 0$.

(2) For $I(x) = \epsilon \ll \frac{1}{2}$, inversion is a little harder; since we'll need to solve the transcendental equation:

$$\epsilon = \frac{1}{\sqrt{2\pi}} \frac{1}{x} e^{-x^2/2}.$$

For $\epsilon \to 0$, the solution to this equation will typically occur for large values of x. taking logs of both sides:

$$\log(\epsilon) = -\log(\sqrt{2\pi}) - \log(x) - x^2/2$$

when $x \gg 1$, the $\log(\sqrt{2\pi})$ and $\log(x)$ terms are $\ll x^2$.

$$\Rightarrow \boxed{x = 2\sqrt{\log\left(\frac{1}{\epsilon}\right)} \qquad \left(\text{when } \epsilon \ll 1\right)}$$